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Weyl quantization and star products

D. Bowes, K.C. Hannabuss *

Mathematical Institute, 24-29 St. Giles', Oxford OX1 3BJ, UK

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Abstract

Many non-linear classical mechanical systems arise as the symplectic reductions of linear systems. The star products on the corresponding quantized algebras can be derived from the Weyl–Moyal product on the algebras of the linear systems. An algebraic approach to Berezin quantization is sketched.

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1. Introduction

It is possible to discern two broad strands amongst algebraic theories of quantization. The traditional methods of Weyl, Moyal, Husimi, Pool, Fock, Cook, Segal and Bargmann [W,M,Hus,P,C,B] are straightforward, precise and rigorous but apply most readily to linear systems (see also [BC,Co] for some less straightforward applications). Newer techniques such as star products and deformations [F,Ba,Li], which have acquired a new importance in the theory of quantum groups, are less restricted, but, despite their beguiling simplicity, their application to particular situations is not always straightforward. The main purpose of this paper is to exploit ideas of geometric quantization to build a bridge between the two approaches. The key idea is that most of the symplectic manifolds which one wishes to quantize are actually symplectic reductions of linear systems. For a linear system one may construct a C^*-algebra of functions using the ideas

* Corresponding author. Fax: 01865 277803.

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of Weyl and Moyal. This can then be pulled back to functions on the original manifold to provide a star product algebra. In spirit this linearization programme is analogous to that relating to geometric and algebraic K-theory, which exploits the possibility of imbedding any vector bundle with connection into a trivial bundle with flat connection.

To be a little more precise, let M be a symplectic manifold with symplectic form ω, and let G be a Lie group of symplectic diffeomorphisms of M. Suppose further that the action is Hamiltonian, that is, for each X in the Lie algebra $\mathfrak{g}$ of G there exists a function ϕ_X on M such that the vector field ξ_X associated to X is dual to the form $\mathrm{d}\phi_X$, that is $\omega(\xi_X, \cdot) = \mathrm{d}\phi_X$. We recall that the moment map Φ from M to the dual Lie algebra $\mathfrak{g}^*$ is defined by setting $\Phi(m)$ to be the linear functional taking X to $\phi_X(m)$ for each m in M [GS1,Wo,So]. Sometimes when the action of G on M can be extended to an action of its complexification $G_{\mathbb{C}}$, it is also useful to define a complex moment map from M to the complexified dual algebra $\mathfrak{g}^*_{\mathbb{C}}$.

Our treatment of star products is closest to that of Fronsdal [F], which seeks to equip a subspace $\mathcal{T}(M)$ of the C^∞-functions on a symplectic manifold M with an associative product. (It has been shown that the ordinary pointwise product does have deformations in this case [dWiL].) This star product satisfies rules which relate it to the usual pointwise product on $C^\infty(M)$ and to the Poisson bracket, but the actual construction is rather formal, unless M is the homogeneous space for a Lie group G. In that case the star product is determined by an 'exponential map', that is a $\mathcal{T}(M)$-valued distribution $g \mapsto E_g$ on the group G satisfying the conditions

$$E_{gh} = E_g \star E_h, \tag{F1}$$

$$E_{ghg^{-1}} = g \cdot E_h, \tag{F2}$$

where $(g \cdot f)(m) = f(g^{-1}m)$ for any function or distribution on M, and the normalization conditions

$$E_1 = 1, \tag{F3}$$

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t}(E_{\exp(tX)})(m) = \phi_X(m) = \Phi(m)(X) \tag{F4}$$

for all $X \in g$ and $m \in M$, where the deformation parameter $\hbar$ is regarded as a constant for a given algebra. (The factor of i appears because we use the mathematicians' convention that the elements of Lie algebras are skew-adjoint rather than self-adjoint.) We shall make some minor modifications of these conditions later. If the smoothed functions

$$\int_G f(g) E_g \, \mathrm{d}g$$

are dense in $\mathcal{T}(M)$ then (F1) can be used to define the star product, and (F4) serves to put the star product into the right relationship with the Poisson bracket.

In the linear case, when M is a real vector space we may identify tangent vectors ξ and η with elements of M. If s is a non-singular skew symmetric bilinear form on M

then we may give M a symplectic form $\omega(\xi, \eta) = s(\xi, \eta)$. If the group G acts linearly on M then the vector $\xi_X(m)$ is identified with $X \cdot m$, and it is easy to check that we may take $\phi_X(m) = \frac{1}{2}s(X \cdot m, m)$, since then $\mathrm{d}\phi_X(m) = s(X \cdot m, \cdot) = \omega(\xi_X, \cdot)$. We shall see in Section 2 that the Fourier transforms of the metaplectic distributions S_g, defined for $g \in S_p(M)$ in [KCH1], provide an exponential map whose associated star product is the Moyal product, provided that the above axioms are slightly modified.

This in turn can be used to furnish an exponential map and a star product for many other symplectic manifolds obtained from M by symplectic reduction. (For any element ξ in g^* we let G_ξ^0 to be the connected component of the stabilizer of ξ under the coadjoint action of G. It can be shown that $M_\xi = \Phi^{-1}(\xi)/G_\xi^0$ is a symplectic manifold called the *reduction* of M at ξ [MW,Wo,LM].) The task is simplified by the fact that most of the interesting reductions are defined by quadratic constraints, and these can be expressed directly in terms of the moment map.

The simplest and best known example is provided by the constraint

$$|z|^2 = 1$$

on vectors $z \in \mathbb{C}^n$, with the imaginary part of the inner product as symplectic form. This can be derived from the moment map for the group of multiplications by scalars of modulus 1, since

$$\mathrm{Im}\left(\frac{\mathrm{d}}{\mathrm{d}t}\langle z, \mathrm{e}^{\mathrm{i}t\theta} z\rangle\right) = \theta|z|^2,$$

and, being abelian, the whole group stabilizes the constraint. The symplectic quotient by this action is the projective space $\mathbb{CP}^{n-1}$ (see, for example, [FCK]). We shall see that the cotangent bundle T^*G of any classical matrix group G can be obtained similarly as a reduction of a symplectic space, as can any coadjoint orbit in the dual Lie algebra $\mathfrak{g}^*$. The orbits of highest weight vectors in G-modules for semi-simple Lie groups G provide another class of example, this time using complex moment maps, as do some symmetric spaces.

The additional ingredient comes from the fact that in linear spaces one has a duality theory for pairs of mutually centralizing subgroups of the symplectic group [Ho1,Ho2,Ho3]. Whenever H is another group whose action commutes with that of G then H also acts on the reduction. We can, therefore, reduce using the moment map of G to obtain a symplectic H-manifold, and vice versa.

Star products on symmetric spaces and on Kähler manifolds have been investigated in two interesting series of papers by Moreno and coworkers [Mo1–7], and by Cahen, Gutt and Rawnsley [RCG,CGR1,CGR2,CGR3], respectively. These have uncovered a wealth of subtle details about the asymptotic behaviour, in particular, and in Section 13 we outline how our approach links with them. (We would like to thank the referee for drawing these to our attention.)

2. Moyal product

We shall start by reviewing the classical ideas of Weyl and Moyal which construct the associative star product for linear symplectic spaces. We shall, with minor modifications, largely follow the exposition of Pool [P]. Let V be a $2n$-dimensional real vector space with symplectic form s. For any constant $\hbar$ the function $\sigma(x, y) = \exp(\mathrm{i}s(x, y)/2\hbar)$ defines a projective multiplier on the additive group of V. Let $\mathcal{S}(V)$ denote the Schwartz functions on V. Using the multiplier σ we may define a twisted convolution product:

$$(f_1 * f_2)(v) = \int_V f_1(v-u) f_2(u) \sigma(v-u, u)^{-1}\,\mathrm{d}u = \int_V f_1(v-u) f_2(u) \exp(\mathrm{i}s(u, v)/2\hbar)\,\mathrm{d}u.$$

Together with the involution $f^*(v) = \overline{f(-v)}$, this product gives $\mathcal{S}(V)$ the structure of a $*$-algebra.

There is a natural Fourier transform on $\mathcal{S}(V)$ given by

$$\mathcal{F}f(v) = \int_V f(u) \frac{\sigma(v, u)}{\sigma(u, v)}\,\mathrm{d}u = \int_V f(u)\sigma(v, 2u)\,\mathrm{d}u = \int_V f(u) \exp(\mathrm{i}s(v, u)/\hbar)\,\mathrm{d}u.$$

This transform differs from that generally used, though the two are equivalent. The first form of the integral is more suitable to generalization, but will not be needed here. We have not bothered to make the transform unitary since this form is better adapted to the normalization of the star product. It is worth noting that this Fourier transform is within a scalar self-inverse, since $\mathcal{F}^2 f = |\Lambda|^{-2} f$, where $\Lambda = (2\pi \mathrm{i}\hbar)^{-n}$ (the factor of i being included for the later convenience).

The Moyal product $\phi \star \psi$ of two functions in $\mathcal{S}(V)$ is defined by

$$\phi \star \psi = \mathcal{F}(\mathcal{F}^{-1}\phi * \mathcal{F}^{-1}\psi),$$

or equivalently

$$\phi \star \psi = |\Lambda|^4 \mathcal{F}(\mathcal{F}\phi * \mathcal{F}\psi).$$

This reduces to the explicit form

$$(\phi \star \psi)(v) = \left(\frac{1}{\pi}\right)^{2n} \int_V \phi(v + \sqrt{\hbar}u)\psi(v + \sqrt{\hbar}w)\mathrm{e}^{-2\mathrm{i}s(u,w)}\,\mathrm{d}u\,\mathrm{d}w.$$

Asymptotic expansion of the integral about $\hbar = 0$ shows that it is approximately the same as the pointwise product, but with a correction proportional to the Poisson bracket of ϕ and ψ computed using the symplectic form s and identifying $T_v^* V$ with V:

$$(\phi \star \psi)(v) = \phi(v)\psi(v) + \tfrac{1}{2}\mathrm{i}\hbar s(\mathrm{d}\phi(v), \mathrm{d}\psi(v)) + \mathrm{O}(\hbar^2).$$

The involution is likewise defined by

$$\mathcal{F}(f^*) = (\mathcal{F}f)^*.$$

3. Linear exponential distributions

The symplectic group $Sp(V)$ which acts linearly on V preserving s, gives rise to an action $S(V)$ defined by $(g \cdot f)(v) = f(g^{-1}v)$. This action gives rise to the well-known calculus of pseudo-differential operators (see, for example [Fo]). For our purposes it will be more convenient to follow [KCH1] where it was shown that to each $g \in Sp(V)$ is associated a tempered distribution $S_g \in \mathcal{S}'(V)$ such that, for any $f \in \mathcal{S}(V)$, both $S_g * f$ and $f * S_g$ lie in $S(V)$ and

$$S_g * f = (g \cdot f) * S_g.$$

The distribution S_g is unique up to scalar multiples and can be normalized so that $S_g * S_h = \alpha(g,h)S_{gh}$, where α is a ± 1-valued on $Sp(V)$. Moreover, the distribution is unitary in the sense that

$$S_g^* = S_{g^{-1}}.$$

The proof of these facts is elementary and depends only on explicit solution of the functional equation. When $g \in Sp(V)$ fixes only the zero vector S_g can be written explicitly as

$$S_g(v) = \Lambda \det(g-1)^{-1/2} \exp\left(\mathrm{i}s\left(v, \left(\frac{g+1}{g-1}\right)v\right) \Big/ 4\hbar\right).$$

In general S_g is supported on $(g-1) \cdot V$, where a similar expression holds, to within a phase factor.

It is also possible to find distributions corresponding to transformations in the affine symplectic group, which is the semi-direct product of $Sp(V)$ with the vector group of translations, $Asp(V) = Sp(V) \bowtie V$. This can be achieved by taking $S_a(v) = \delta(v-a)$ for v in V, and setting $S_{A,a} = S_a * S_A$ for general elements, which gives the explicit formula

$$S_{(A,a)}(v) = \mathrm{e}^{-\mathrm{i}s(a,v)/2\hbar} S_A(v-a) = \mathrm{e}^{-\mathrm{i}\phi_a(v)/2\hbar} S_A(v-a),$$

where $\phi_a(v) = s(v,a)$ is the moment map for a translation.

As noted in [HaHe] there is a link between the Fourier transform and convolution with S_{-1}. (See also [Mo?], which uses this as the basis of a quantization procedure in symmetric spaces.)

Theorem 3.1. *The distribution $\mathcal{F}f$ can be expressed as*

$$(\mathcal{F}f)(v) = 2^n \Lambda^{-1}(S_{-1} * f)(2v),$$

and, in particular,

$$(\mathcal{F}S_g)(v) = 2^n \Lambda^{-1} S_{-g}(2v).$$

Proof. Since $S_{-1}(v) = 2^{-n}\Lambda$ we have

$$(S_{-1} * f)(v) = 2^{-n}\Lambda \int_V f(u)\mathrm{e}^{\mathrm{i}s(v,u)/2\hbar}\,\mathrm{d}u = 2^{-n}\Lambda\mathcal{F}f(\tfrac{1}{2}v),$$

whence the general result follows. Since $\alpha(-1, g) = 1$, we have, in particular,

$$S_{-g}(v) = 2^{-n}\Lambda\mathcal{F}S_g(\tfrac{1}{2}v),$$

so that

$$\mathcal{F}S_g(v) = 2^n\Lambda^{-1}(S_{-1} * S_g)(2v) = 2^n\Lambda^{-1}(S_{-g})(2v). \qquad \square$$

Now we know that for $g \in Sp(V)$,

$$2^n\Lambda^{-1}S_{-g}(2v) = 2^n\det(g+1)^{-1/2}\exp\left(\mathrm{i}s\left(v, \left(\frac{g-1}{g+1}\right)v\right)\Big/\hbar\right).$$

The Cayley transform which sends $g \in Sp(V)$ to $X(g) = (g+1)^{-1}(g-1)$ can be regarded as a map from the elements g in $Sp(V)$ for which $g+1$ is non-singular to its Lie algebra $sp(V)$, and with this interpretation we have

$$\mathcal{F}S_g = \det(1 - X(g))^{1/2}\exp(-2\mathrm{i}\phi_{X(g)}(v)/\hbar).$$

This result extends to $g = (A, a) \in Asp(V)$, giving

$$\mathcal{F}S_{(A,a)} = \det(1 - X(A))^{1/2}\exp(-\mathrm{i}(2\phi_{X(A)}\big(v + \tfrac{1}{2}a\big) + \phi_a(v))/\hbar).$$

This provides an obvious candidate for the exponential distribution since we have the following result.

Theorem 3.2. *The distribution $E_g = \mathcal{F}S_g$ satisfies the following identities:*

$$E_g \star E_h = \alpha(g,h)E_{gh}, \qquad E_{ghg^{-1}} = \alpha(g,h)\alpha(gh, g^{-1})(g \cdot E_h), \qquad E_1 = 1$$

for all g and $h \in Asp(V)$ and $X \in \mathfrak{asp}(V)$.

Proof. The Fourier transform of S_g is E_g so that by the definition of the Moyal product we have

$$(E_g \star E_h) = \mathcal{F}(S_g * S_h) = \alpha(g,h)\mathcal{F}S_{gh} = \alpha(g,h)E_{gh},$$

from which the first identity follows immediately.

Similarly, the second equation follows on taking the Fourier transform of the identity

$$S_{ghg^{-1}} = \alpha(g,h)\alpha(gh, g^{-1})(g \cdot S_h).$$

By the previous result we know that

$$E_1(v) = 2^n\Lambda^{-1}S_{-1}(2v) = 1,$$

giving the required normalization. $\square$

Theorem 3.3. *For any $g \in Sp(V)$ and $Y \in \mathfrak{sp}(V)$*

$$E_g \star \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} E_{\exp(tY)}|_{t=0} = E_g \cdot (\phi_Y(2(g+1)^{-1}v) + \tfrac{1}{2}\mathrm{i}\hbar\, \mathrm{tr}((g+1)^{-1}Y)).$$

In particular, for any $Y \in \mathfrak{sp}(V)$

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} E_{\exp(tY)}|_{t=0} = \phi_Y(v).$$

Proof. We differentiate the identity

$$E_g \star E_{\exp(tY)} = \alpha(g, \exp(tY)) E_{g\exp(tY)}.$$

with respect to t, using the fact that for small t the discrete multiplier must be identically 1, to obtain

$$E_g \star \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} E_{\exp(tY)} = \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} E_{g\exp(tY)}.$$

We then note that

$$\begin{aligned}
&\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} E_{g\exp(tY)}|_{t=0}\\
&= \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} 2^n \det(g\mathrm{e}^{tY}+1)^{-1/2} \exp(\mathrm{i}s(v, (g\mathrm{e}^{tY}+1)^{-1}(g\mathrm{e}^{tY}-1)v)/\hbar)|_{t=0}\\
&= \mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} \det(\tfrac{1}{2}(g\mathrm{e}^{tY}+1))^{-1/2} \exp(\mathrm{i}s(v, (1-2(g\mathrm{e}^{tY}+1)^{-1})v)/\hbar)|_{t=0}\\
&= (-2s(v, (g+1)^{-1}gY(g+1)^{-1}v) - \tfrac{1}{2}\mathrm{i}\hbar\, \mathrm{tr}((g+1)^{-1}gY))E_g.
\end{aligned}$$

Since the elements of $\mathfrak{sp}(V)$ have vanishing trace this can be rewritten as

$$(\phi_Y(2(g+1)^{-1}v) + \tfrac{1}{2}\mathrm{i}\hbar\, \mathrm{tr}((g+1)^{-1}gY))E_g,$$

whence the main result now follows. The second result follows on setting $g = 1$ in the earlier identities and using the vanishing trace condition again. □

Note. With rather more effort one can produce a similar result for $Asp(V)$. These last two results combine to show that E_g provides an explicit form of the exponential map in the linear situation, provided we are prepared to tolerate a few technical differences from the properties postulated by Fronsdal. For a start our $E_g(v)$ is a distribution in v, rather than a function, although it happens to be a function whenever -1 is not in the spectrum of g. Moreover, $E_g(v)$ is not actually a distribution in g but the section of a line bundle $\mathcal{L}$. (This can be seen by considering the behaviour of S_g as g tends to an element which has fixed points, so that $g-1$ is singular. In fact, $\mathcal{L}$ is a pullback of the Maslov bundle. One sees by Fourier transformation that the distribution is well-defined albeit with a transition function [KCH2].) Finally, the metaplectic multiplier α, does not appear at all in the usual star product formulae. This is presumably because its restriction to many of the usual subgroups is trivial and, in any case, being discrete, it does not appear in the Lie algebraic approach which forms the basis of most the accounts.

Theorem 3.4. *The star product of moment functions satisfies the identity*

$$\phi_X \star \phi_Y = \phi_X\phi_Y + \tfrac{1}{2}\mathrm{i}\hbar\phi_{[X,Y]} + \tfrac{1}{8}\hbar^2\,\mathrm{tr}(XY).$$

The star product of any function $f \in \mathcal{S}(V)$ with a constant c is given by $c \star f = cf$.

Proof. We set $g = \exp(uX)$ in Theorem 3.3, differentiate the outermost terms of the main identity with respect to u at $u = 0$, and multiply by $\mathrm{i}\hbar$. Using the special case, this gives for the star product $\phi_X \star \phi_Y$,

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}u}E_{\exp(uX)}\cdot(\phi_Y(2(\mathrm{e}^{uX}+1)^{-1}v) + \tfrac{1}{2}\mathrm{i}\hbar\,\mathrm{tr}((\mathrm{e}^{-uX}+1)^{-1}Y)).$$

Now

$$\frac{\mathrm{d}}{\mathrm{d}u}(\mathrm{e}^{uX}+1)^{-1} = -(\mathrm{e}^{uX}+1)^{-1}X\mathrm{e}^{uX}(\mathrm{e}^{uX}+1)^{-1},$$

so that the derivative of ϕ_Y is

$$\begin{aligned}&-2\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}u}s((\mathrm{e}^{uX}+1)^{-1}v, Y(\mathrm{e}^{uX}+1)^{-1}v)|_{u=0}\\ &\quad= \tfrac{1}{4}\mathrm{i}\hbar(s(Xv,Yv)+s(v,YXv)) = -\tfrac{1}{4}\mathrm{i}\hbar s(v,(XY-YX)v) = \tfrac{1}{2}\mathrm{i}\hbar\phi_{[X,Y]}.\end{aligned}$$

The trace term similarly gives the derivative

$$\mathrm{i}\hbar\frac{\mathrm{d}}{\mathrm{d}u}(\tfrac{1}{2}\mathrm{i}\hbar\,\mathrm{tr}((\mathrm{e}^{-uX}+1)^{-1}Y)) = \tfrac{1}{8}\hbar^2\,\mathrm{tr}(XY).$$

Substituting these into the formula for the derivative we get

$$\phi_X \star \phi_Y = \phi_X(v)\phi_Y(v) + \tfrac{1}{2}\mathrm{i}\hbar\phi_{[X,Y]} + \tfrac{1}{8}\hbar^2\,\mathrm{tr}(XY).$$

As far as the constants are concerned we note that

$$\mathcal{F}^{-1}(E_1 \star f) = S_1 * \mathcal{F}^{-1}f = \mathcal{F}^{-1}f,$$

since S_1 is a delta function. We can therefore deduce that $1 \star f = f$, and the general result follows by linearity. □

By replacing g by $g\exp(uX)$ in Theorem 3.3 it is also possible to generalize this to obtain a formula for $E_g \star \phi_X \star \phi_Y$. The same technique may then be applied to obtain formulae for star products of any number of moment maps. For simplicity, except where explicitly mentioned, we shall restrict our discussion to the case of linear symplectic transformations throughout the rest of this paper.

4. Coproducts

One of the grounds for the resurgence of interest in deformation theory has been the development of quantum group theory. In that context the twisted convolution and Moyal bracket described in the previous sections have a particularly interesting property, which we shall describe in this section.

A quantum group is a Hopf algebra, so that there is a coproduct as well as a product. The product on a group algebra is usually the convolution (or twisted convolution) product, and the coproduct is the dual of pointwise multiplication. Now the star product is a deformation of pointwise multiplication, so this suggests that its dual would make an interesting new coproduct. More specifically (using a conjugate linear pairing)

$$\begin{aligned}\langle \delta f, \phi \otimes \psi \rangle &= \langle f, \phi \star \psi \rangle \\ &= \pi^{-2n} \int \overline{f(v)}\phi(v + \sqrt{\hbar}u)\psi(v + \sqrt{\hbar}w)\mathrm{e}^{-2\mathrm{i}s(u,w)}\,\mathrm{d}u\,\mathrm{d}w\,\mathrm{d}v \\ &= (\hbar\pi)^{-2n} \int \overline{f(v)}\mathrm{e}^{-2\mathrm{i}s(x-v,y-v)/\hbar}\phi(x)\psi(y)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}v.\end{aligned}$$

This means that

$$\begin{aligned}(\delta f)(x, y) &= (\hbar\pi)^{-2n} \int f(v)\mathrm{e}^{2\mathrm{i}s(x-v,y-v)/\hbar}\,\mathrm{d}v \\ &= (\hbar\pi)^{-2n}\mathrm{e}^{2\mathrm{i}s(x,y)/\hbar} \int f(v)\mathrm{e}^{2\mathrm{i}s(y-x,v)/\hbar}\,\mathrm{d}v \\ &= \pi^{-2n} \int f(\tfrac{1}{2}(x + y) + \hbar z)\mathrm{e}^{2\mathrm{i}s(y-x,z)}\,\mathrm{d}z.\end{aligned}$$

In this form it is easy to see that as $\hbar \mapsto 0$ this tends to the limit

$$(\delta f)(x, y) = f(\tfrac{1}{2}(x + y))\delta(y - x),$$

which is the usual coproduct. Alternatively we may rewrite the earlier expression as

$$(\delta f)(x, y) = (2\pi)^{-2n}(\mathcal{F}f)(2(y - x))\mathrm{e}^{2\mathrm{i}s(x,y)/\hbar}.$$

One would also like to know whether a coproduct is equivalent to its transpose. In the case of the twisted convolution of Schwartz functions this has a rather unexpected answer. Now any coproduct on $\mathcal{S}(V)$ maps it to $\mathcal{S}(V) \otimes \mathcal{S}(V) \cong \mathcal{S}(V \oplus V)$. The transposition map $I : (u, v) \mapsto (v, u)$ on $V \oplus V$ is clearly symplectic with respect to the form $s \oplus s$. It can therefore be implemented by a tempered distribution $\tilde{S}_I \in \mathcal{S}'(V \oplus V)$. That is, writing $*$ for the twisted convolution on $\mathcal{S}(V \oplus V)$, we have

$$\tilde{S}_I * f = (f \circ I) * \tilde{S}_I,$$

which shows that interchange of the two factors in $\mathcal{S}(V) \otimes \mathcal{S}(V)$ is an inner automorphism. This in turn means that every coproduct can be turned to its transpose by inner automorphism with $\tilde{S}_I$.

We can, of course, give an explicit formula for $\tilde{S}_I$. It is supported on

$$(I-1)(V \oplus V) = \{(z, -z) \in V \oplus V : z \in V\},$$

where it takes the form

$$\tilde{S}_I((I-1)w) = c_I \exp(\tfrac{1}{2}\mathrm{i}(s \oplus s)(w, (I-1)w))$$

for some constant c_I and $w \in V \oplus V$. Now, by antisymmetry we have

$$(s \oplus s)(w, (I-1)w) = (s \oplus s)(w, Iw),$$

which can be rewritten as

$$(s \oplus s)(I^2 w, Iw) = (s \oplus s)(Iw, w) = -(s \oplus s)(w, Iw),$$

and so vanishes. Consequently $\tilde{S}_I$ is constant on $(I-1)(V \oplus V)$. (Much more detailed discussion of star products in the context of quantum groups can be found in [Mo6,7], and the works cited there.)

5. Metaplectic representation

So far we understood how to construct star products and exponential maps on symplectic vector spaces, and reduce the vector space to a symplectic manifold. Our next task is to provide a practical description of how the reduction process affects the functions, distributions and products. There is more than one way to approach this, but for our purposes it will be useful to review some facts about the metaplectic representation.

The Stone–von Neumann theorem tells us that there is, up to equivalence, a unique irreducible σ-representation, W, of the additive group of V. This can also be extended to a $*$-representation of the twisted convolution algebra of Schwartz functions by defining

$$W(f) = \int_V W(v) f(v)\,\mathrm{d}v.$$

By transposition a similar definition works for distribution f.

Since any $g \in Sp(V)$ preserves σ, the map $v \mapsto W(g \cdot v)$ also defines an irreducible σ-representation and so it must be equivalent to W; that is there exists a unitary operator, $U(g)$ such that

$$U(g) W(v) U(g)^{-1} = W(g \cdot v).$$

In fact, with our earlier notation it is easy to see that

$$U(g) = W(S_g) = \int_V S_g(v) W(v)\,\mathrm{d}v$$

will implement the equivalence, from which it is easy to see that U is a projective representation of $Sp(V)$ with multiplier α. This projective representation is known as the metaplectic representation of $Sp(V)$. Conversely if $U(g)$ is known then the distribution can be recovered from the identity

$$\int_V \overline{S_g(v)} f(v)\,\mathrm{d}v = |\Lambda| \mathrm{tr}(U(g)^* W(f)),$$

which holds for any Schwartz function $f \in \mathcal{S}(V)$ by the Plancherel theorem. Formally this means that $S_g(v) = |\Lambda| \mathrm{tr}(U(g)W(v)^*)$.

Bearing in mind that $|\Lambda|$ is the formal dimension, the relationship between S_g and the metaplectic representation is the anologue of Fronsdal's character formula for his exponential distribution [F, Eq. (97)].

Theorem 5.1. *The exponential function is related to the character of the metaplectic representation by*

$$\int_V E_g(v)\,\mathrm{d}v = |\Lambda|^{-1} \mathrm{tr}(U(g)).$$

Proof. We have by definition

$$\begin{aligned}\int_V E_g(v)\,\mathrm{d}v &= \mathcal{F}E_g(0) = \mathcal{F}^2 S_g(0)\\ &= |\Lambda|^{-2} S_g(0) = |\Lambda|^{-1} \mathrm{tr}(U(g)).\end{aligned}$$

□

6. Dual pair of subgroups

Let us now suppose that G is a reductive subgroup of $Sp(V)$ and so is its centralizer, G'. Suppose further that $G'' = G$. It is known that if one of G and G' is compact then there is a bijection between the irreducible components of $U|_G$ and $U|_{G'}$ [Ho1]. More precisely, to each irreducible direct summand D of $U|_G$ there exists a unique irreducible summand D' of $U|_{G'}$ such that the restriction of the metaplectic representation to GG' can be written as

$$U|_{GG'} = \bigoplus D \otimes D',$$

where each D occurs only for a single D'. (The elements of $G \cap G'$, being central, act as scalars so can be put in either term of the tensor product.) This decomposition can be derived from the fact that the distributions S_g with $g \in G$ are G'-invariant and they generate all G'-invariant tempered distributions.

This result can be regarded as the quantum analogue of reduction, since it enables one to specify the behaviour of the system under the action of G'. (If the system transforms under the representation D' of G' then one needs only to work with the corresponding representation D of G, just as symplectic reduction at $\alpha \in g'$ picks out a symplectic G-manifold M_α).

Dual pairs occur naturally whenever one takes the space V to be the space $\mathcal{L}(Y, Z)$ of linear operators between two finite-dimensional Hilbert spaces Y and Z, with the symplectic form

$$s(x, y) = \mathrm{Im}(\mathrm{tr}(x^* y)).$$

The unitary groups $U(Z)$ and $U(Y)$ act on the left and right of V and each is the centralizer of the other. Another simple class of examples is generated by taking a real inner product space (Y, B) and a symplectic space (Z, s_0), both finite-dimensional, and forming $V = Y \otimes Z$ with $s = B \otimes s_0$ as symplectic form. In this case $O(Y)$ and $Sp(Z)$ form the dual pair. In the special case when Y is one-dimensional so that $V = Z$, this tells us that the metaplectic representation splits into just two irreducibles associated to the two irreducibles of $O(Y) = \mathbb{Z}_2$. There are other examples of dual pairs, and the reductive dual pairs can be classified.

Writing $\mathcal{H}_D$ and $\mathcal{H}_{D'}$ for the representation spaces of D and D' we see that the representation space $\mathcal{H}$ of W decomposes as

$$\mathcal{H} = \bigoplus \mathcal{H}_D \otimes \mathcal{H}_{D'}.$$

The G'-invariant operators on this space are just those of the form

$$A = \bigoplus A_D \otimes 1_{D'},$$

where $1_{D'}$ is the identity operator on $\mathcal{H}_{D'}$, and A_D is any operator on $\mathcal{H}_D$.

It is known that every bounded operator on $\mathcal{H}$ can be expressed in the form

$$W(F) = \int_V F(v) W(v)\, \mathrm{d}v$$

for some tempered distribution F [Lu], suggesting that the G invariant distributions will decompose into a direct sum of matrix algebras $\mathcal{L}(\mathcal{H}_{D'})$. The individual components can be picked out using the central idempotents in algebra. For example, when G' is abelian so that $G' \subseteq G$ we need only specify the representation D'. If χ_D is the character of D' then the projection onto the component is

$$P_D = \int_{G'} \overline{\chi_D(h)} U(h)\, \mathrm{d}h$$

and the corresponding distributional projection is

$$p_D = \int_{G'} \overline{\chi_D(h)} S_h\, \mathrm{d}h = A \mathrm{tr}(P_D W(v)^*).$$

Since in this case G' is compact, the projection P_D has a finite rank and p_D is actually a Schwartz function. Duality means $p_D = p_{D'}$. There is an extensive discussion of duality in the context of geometric quantization in [GS1,GS2].

7. Contangent bundles, coadjoint orbits and symmetric spaces

Before discussing how the linear results can be pulled back to reductions, it is helpful to consider some examples to show the sort of systems which can be covered by this procedure. Often the symplectic space has the form $V = X \oplus X^*$, where X^* is the dual of X, and the symplectic form is given by

$$s((x,\xi),(y,\eta)) = \eta(x) - \xi(y).$$

Suppose that one has a symmetric bilinear map B from $X \times X$ to some space L. For $\lambda \in L^*$, consider the linear transformation

$$g(\lambda) : (x,\xi) \mapsto (x, \xi + \lambda(B(x,\cdot))),$$

which are automatically symplectic, and together they define a nilpotent subgroup G_B of $Sp(V)$. The Lie algebra is also parametrized by L^* and a typical element has the form

$$(x,\xi) \mapsto (0, \lambda(B(x,\cdot))),$$

from which it follows that the moment map is defined by

$$\phi_\lambda(x,\xi) = s((x,\xi),(0,\lambda(B(x,\cdot)))) = -\tfrac{1}{2}\lambda(B(x,x)).$$

Fixing the moment map is equivalent to fixing $B(x,x)$. Let us denote by S_ξ the level surface of x for which $B(x,x) = \xi$. Since G_B is abelian, each $g(\lambda)$ stabilizes any chosen point, ξ, in the dual Lie algebra. The action of G_B can change ξ to any element of the form $\xi + \lambda(B(x,\cdot))$. Now for fixed x all such elements give the same value to elements y in the subspace B-orthogonal to x, which, away from singular points x, can be identified with the tangent space $T_x S_\xi$ to the level surface. We may thus identify the orbit of ξ under G_B with the dual $T_x^* S_\xi$ of $T_x S_\xi$. Thus the reduction process gives the cotangent bundle $T^* S_\xi$.

There are numerous interesting examples of this. The simplest situation is, of course, that of a real-valued quadratic form, when S_ξ is a quadratic hypersurface in X. Even in this situation there can be interesting variations. If for example (X, B) is a Lorentzian space (that is B has signature $(1,q)$), and we identify M^* with M using B, we have

$$B(\xi + tx, \xi + tx) = B(\xi,\xi) + 2tB(x,\xi) + t^2 B(x,x) = B(\xi,\xi) + 2tB(x,\xi) + t^2.$$

Since $B(x,x) = 1$, we may choose a basis with x as one element from which we see that $B(\xi,\xi) < B(\xi,x)^2$, from which it follows that, with a suitable choice of t we can arrange that $B(\xi + tx, \xi + tx) = 0$. Indeed, there is a unique point on each orbit such that ξ lies on the null cone. Now, we already knew that the reduction would give the cotangent bundle to the hyperboloid $B(x,x) = 1$, so this tells us that this cotangent bundle can be identified with the product of the hyperboloid and the null cone. It is, in fact, well known that the null cone plays a similar role in harmonic analysis on hyperbolic space to that of momentum space in a flat space.

A more complicated situation arises when one takes $X = \mathcal{L}(Y, Z)$, the space of linear transformations between two finite-dimensional real inner product spaces Y and Z. Then

$B(x, y) = \frac{1}{2}(x^*y+y^*x)$ defines a bilinear form from X to $\mathcal{L}(Y)$. The level surface on which $B(x, x) = 1_Y$ is just the Stiefel manifold $O(Y, Z)$ of isometric imbeddings from Y into Z. (We shall assume that $\dim Y \leq \dim Z$, so that this is non-empty.) Reduction thus enables us to construct its cotangent bundle. In particular, when $Y = Z$ this gives the cotangent bundle $T^*O(Y)$ and, at the other extreme, when $Y = \mathbb{R}$ we have $\mathcal{L}(Y, Z) = Z$ and reduction gives the cotangent bundle of the unit sphere in Z. More generally, the orthogonal group of Z acts transitively on the isometries (by composition). An isometry I is fixed by those elements of $O(Z)$ which acts as the identity on IY, so that the stabilizer can be identified with the subgroup $O(IY^{\perp})$ and $S_1 = O(Z)/O(IY^{\perp})$.

To interpret such examples in terms of dual pairs it is useful to note that the constraint $x^*x = 1_u$ is $O(Y)$ invariant. This suggests that we might use the semi-direct product subgroup $O(Y) \bowtie \mathcal{L}(Y)^*$, rather than just $\mathcal{L}(Y)^*$. It is easy to check that the subgroup of $Sp(X \oplus X^*)$ which centralizes this is $O(Z) \bowtie \mathcal{L}(Z)^*$, giving a pleasingly symmetric dual pair. (This example can be interpreted in terms of the $G - G'$ bimodule X in which the constraint is defined by a rigging in the sense of Rieffel, [Ri].)

Unfortunately, neither of the two subgroups is reductive nor much less compact. However, numerous examples suggest that the theory of dual pairs extends to mutually centralizing non-reductive subgroups of the $Sp(V)$, and even to subgroups of the affine symplectic group $Asp(V) = Sp(V) \bowtie V$ provided the subgroups lift to those which commute in the central extension [KCH2]; but there are no general theorems to assist the analysis. (In fact, if one had a theorem for non-reductive subgroups then one could deduce the result for the affine group by embedding it in $Sp(V \oplus \mathbb{R}^2)$.) Nonetheless, we can verify duality by direct means in this case. The metaplectic representation can be realized on $L^2(X)$. An element (A, α) in the subgroup $O(Y) \bowtie \mathcal{L}(Y)^*$ is represented by

$$(U(A, \alpha)\psi)(x) = e^{i\alpha(A^*x^*xA)/2}\psi(xA).$$

According to Mackey's semi-direct product theory [Ma] this decomposes into a direct integral of primary representations (containing only one type of irreducible) parameterized by the $O(Y)$ orbits of x^*x in $\mathcal{L}(Y) = \mathcal{L}(Y)^{**}$. These orbits can in turn be labelled by the spectrum of the self-adjoint operator x^*x.

Similarly B, β in the centralizer $O(Z) \bowtie \mathcal{L}(Z)^*$ has metaplectic representation

$$(U(B, \beta)\psi)(x) = e^{-i\beta(B^*xx^*B)/2}\psi(B^*x).$$

This time the primary subrepresentations are parameterized by the spectrum of xx^*. Now, the spectrum of xx^* is entirely determined by that of x^*x, since the non-zero eigenvalues are the same. This shows that there is a bijection between the irreducible subrepresentations of $O(Y) \bowtie \mathcal{L}(Y)$ and of $O(Z) \bowtie \mathcal{L}(Z)$ in the metaplectic representation. Moreover, since x is determined up to orthogonal transformations by the fact that it maps the eigenvectors of x^*x to those of xx^*, we see that the metaplectic representations of $O(Y) \bowtie \mathcal{L}(Y)$ and $O(Z) \bowtie \mathcal{L}(Z)$ do generate each other's commutants.

Returning to the reduction process, as a final example of this sort we take $Y = Z \oplus \mathbb{R}$, to obtain $\mathcal{L}(Y, Z) = \mathcal{L}(Z) \oplus Z$. The level surface defined by $P_Z B(x, x) P_Z = P_Z$, the projection onto Z picks out the Euclidean group $E(Z) = O(Z) \times Z$ so that we obtain for the reduction $T^*E(Z)$.

These examples were motivated by [Li], and the construction can clearly be generalized to the other compact matrix groups and symmetric spaces (and many non-compact ones too), and since all are defined by quadratic relations. It is also known that every coadjoint orbit of $O(Z)$ occurs as a reduction of $T^*O(Z)$, [MW], and so can be obtained by reduction of the linear space V. Some of the exceptional groups can be realized as automorphisms of Jordan and Cayley algebras $\mathcal{A}$ [Sc], and so are linear transformations satisfying the inhomogeneous quadratic homomorphism constraints, $g(x \circ y) = g(x) \circ g(y)$ for all x and y in $\mathcal{A}$, where $\circ$ denotes the product.

Three of the four classical series of hermitian symmetric spaces have simple constructions of this kind using a dual pair G, G' with $G \cap G' = K$ to construct the symmetric space G/K. We illustrate this in the case of $Sp(2n, \mathbb{R})/U(n)$, recalling that $U(n) = Sp(2n, \mathbb{R}) \cap O(2n)$. We therefore take $V = \mathcal{L}(\mathbb{R}^{2n})$, with $G = Sp(2n, \mathbb{R})$ acting on the left and $G' = O(2n)$ acting on the right. Taking on basis $\{e_j, j = 1, \ldots, 2n\}$ which is orthonormal with respect to the inner product preserved by $O(2n)$, and letting b be the symplectic form on $\mathbb{R}^{2n}$, we define the symplectic pairing of $S, T \in \mathcal{L}(\mathbb{R}^{2n})$ to be

$$s(S, T) = \sum_{j=1}^{2n} b(Se_j, Te_j).$$

This is independent of the particular orthonormal basis chosen. The moment map sends $X \in \mathfrak{g}'$ to $\phi_X(T) = \sum_j b(Te_j, TXe_j)$, and the inverse image of the linear functional $X \mapsto \sum_j b(e_j, Xe_j)$ is precisely the set of $T \in G$. Our chosen linear functional is stabilized by those $h \in G'$ for which $b(e_j, h^{-1}Xhe_j) = b(e_j, Xe_j)$. Since this is also the same as $b(he_j, Xhe_j)$, we see that h must stabilize b as well as the inner product, that is $h \in G \cap G' = U(n)$. Reversing the roles of $O(2n)$ and $sp(2n, \mathbb{R})$ gives $O(2n)/U(n)$. (Pseudo-orthogonal groups are also allowed, so that one can also obtain the non-compact forms.) The dual pair $U(p+q)$ and $U(p, q)$ acting on the right and left of $\mathcal{L}(\mathbb{C}^{(p+q)})$ with a symplectic form given by the imaginary part of the obvious inner product, similarly lead to $U(p+q)/U(p) \times U(q)$ and $U(p, q)/U(p) \times U(q)$.

8. Highest weight orbits

Another useful class of examples is provided by the following result.

Theorem 8.1. *Let K be a compact semi-simple Lie group. Every integral coadjoint orbit of K in the dual of its Lie algebra can be constructed by reduction of a symplectic vector space.*

Proof. By the techniques of geometric quantization we may associate to each integral coadjoint orbit a unitary representation of K [Wo,GS] on a complex inner product space V. The imaginary part of the inner product gives V a symplectic form. There is a $U(1)$-action on V provided by multiplication by complex numbers of modulus 1, which may be used as in the introduction to construct the projective space $P(V)$. We shall write $[v]$ for the point in the projective space corresponding to the vector v in V. Now it is known that when v is a highest weight vector for the representation of K then the K-orbit of $[v]$ in the projective space is isomorphic, as a symplectic K-manifold, to the coadjoint orbit used [R,GS]. On the other hand it is known that the K-orbit of $[v]$ is defined by a set of quadratic equations, [DF,KCH3]. Indeed, if $\{X_\beta\}$ forms a basis for the Lie algebra and $K^{\beta\gamma}$ denotes the dual Killing form, then the vectors w in the orbit of a highest weight vector satisfy the quadratic equations

$$K^{\beta\gamma} X_\beta w \otimes X_\gamma w = \langle \lambda, \lambda \rangle w \otimes w,$$

where λ is the highest weight and the norm is calculated using the Killing form. The orbit can, therefore, be obtained by a further reduction of projective space. Alternatively the two reductions can be combined into a single reduction of the vector space V. □

Example. Consider the three-dimensional irreducible representation of $SO(3)$ on $\mathbb{C}^3$. The action of $SO(3)$ preserves not only the inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$, but also the complex bilinear dot product $\boldsymbol{u} \cdot \boldsymbol{v}$. The orbit of a highest weight vector is in this case defined by the single equation $\boldsymbol{w} \cdot \boldsymbol{w} = 0$, which can be combined with the normalization condition $\| \boldsymbol{w} \|^2 = 1$ to provide the reduction.

This examples brings to light an important phenomenon. Since we are dealing with real spaces, the constraint $\boldsymbol{w} \cdot \boldsymbol{w} = 0$ automatically forces the conjugate constraint $\boldsymbol{w}^* \cdot \boldsymbol{w}^* = 0$. The Hamiltonian vector fields to which these give rise are $\boldsymbol{w} \cdot \nabla_{\boldsymbol{w}^*}$ and $\boldsymbol{w}^* \cdot \nabla_{\boldsymbol{w}}$, which together with $\boldsymbol{w} \cdot \nabla_{\boldsymbol{w}} - \boldsymbol{w}^* \nabla_{w^*}$ arising from $|\boldsymbol{w}|^2$ generate the Lie algebra of $SL(2, \mathbb{R})$, the centralizer of $O(3)$. This fits into the general scheme for reduction given a pair of dual subgroups. Unfortunately, the general picture is not so simple. The quadratics defining the orbit, together with their conjugates generate too large an algebra, to be useful. However, the solution is obvious: we know that everything in this case is complex, so we work within the complex symplectic group, thus totally avoiding the necessity to introduce the conjugate constraints. (In essence this exploits the connections between complex algebraic geometry and real symplectic geometry developed in [FCK].) The reductions obtained by this procedure are always Kähler manifolds.

Corollary 8.2. *Every integral symplectic K-manifold on which K acts transitively can be obtained by reduction of a symplectic vector space.*

Proof. Every integral transitive symplectic K-manifold is isomorphic to a coadjoint orbit, so that the result follows immediately. □

Remark. By complexifying the Lie algebra this result can be extended to all orbits of semi-simple Lie groups which give rise to highest weight representations. In particular discrete series representations of semi-simple Lie groups can be obtained by reduction of symplectic vector spaces.

There is an interesting class of infinite-dimensional examples provided by the Hirota bilinear form for completely integrable models. It is known that the solution space of many integrable models can be realized as the orbit of certain vectors in the representation space of a loop group (or affine Lie algebra). The Hirota bilinear form characterizes the vectors Ω on the orbit as these for which $\Omega \otimes \Omega$ generates an irreducible cyclic representation of the loop group. This is entirely analogous to the definition of highest weight vectors by quadratic constraints.

9. Non-linear star products

For simplicity, we shall henceforth write $H = G'$ for the centralizer of G. The moment map $\Phi : V \mapsto h^*$ induces a map Φ^* from $\mathcal{S}(h^*)$ to H-invariant Schwartz functions $\mathcal{S}^H(V)$ by $\Phi^* F = F \circ \Phi$. We want a space of C^∞-functions on the orbit $\mathcal{O} \subseteq h^*$ or equivalently on $\Phi^{-1}(\mathcal{O}) \subseteq V$. Usually this is the quotient of $\mathcal{S}^H(V)$ by the star ideal of functions vanishing on $\mathcal{O}$. This ideal is generated by $\Phi_X(v) - D'(X)$ for $X \in h$. When H is abelian this is equivalent to that generated by $\mathcal{F}S_h - \chi_{D'}(h)$ for $h \in H$, and is always equivalent to taking the ideal generated by $p_{D'} - 1$, where $p_{D'} = p_D$ is the distributional projection onto the space where S_h acts as the character $D'(h)$. (Invariance under the involution follows since $p_D = p_D^*$.) Since we want to construct a star product algebra we take the $*$-ideal $\mathcal{I}_D$ generated by these elements under the Moyal product. Our algebra will therefore be $\mathcal{S}_D(V) = \mathcal{S}^H(V)/\mathcal{I}_D$ equipped with the quotient product and involution inherited from the Moyal algebra. (Quotients of a slightly different kind were used in [Mo2]).

As one might expect, the structure of this algebra becomes more transparent when one Fourier transforms back to the twisted convolution algebra. The ideal $\mathcal{I}_D$ is transformed back to $\mathcal{F}^{-1}\mathcal{I}_D$, the principal $*$-ideal generated under twisted convolution by $\delta - p_D$. We therefore have the quotient of the H-invariant functions $\mathcal{S}^H(V)$ by the ideal $\mathcal{S}^H(V) * (\delta - p_D)$, and this automatically inherits a star product from the Moyal product on $\mathcal{S}^H(V)$.

When D' occurs in the metaplectic representation as a discrete summand the projection p_D is itself a function, and we may write

$$\mathcal{S}^H(V) = \mathcal{S}^H(V) * p_D + \mathcal{S}^H(V) * (\delta - p_D).$$

The quotient by $\mathcal{F}^{-1}\mathcal{I}_D$ is then $\mathcal{S}^H(V) * p_D$, which consists of functions $\phi \in \mathcal{S}^H(V)$ such that $\phi = \phi * p_D$. Equivalently when H is abelian, it consists of ϕ such that

$$S_t * \phi = D'(t)\phi$$

for all $t \in H$. This exhibits the quotient algebra as the algebra induced from by D'. The above procedure could be generalized to more general H by looking at $\mathcal{L}(\mathcal{H}'_D)$-valued functions ϕ satisfying a similar equivariance condition.

Note. The construction of representations of star product algebras by Rieffel's C^*-algebra inducing has been discussed by Hennings in [He], and we do not intend to go into any detail here, except to note that combined with the present approach it reproduces the results of Landsman [La] in the case of reduction of constrained linear systems.

Theorem 9.1. *Assume that H and G form a compact dual pair. The star product algebra constructed $\mathcal{S}^H(V)/\mathcal{I}_D$ is isomorphic to the algebra of operators $\mathcal{L}(\mathcal{H}_D)$.*

Proof. We know that $P_D = W(p_D)$ is the projection which picks the D'-isotypic component from the metaplectic representation, and by duality this is just $D \otimes D'$. The Weyl quantization $Q_W = W \circ \mathcal{F}^{-1}$ maps the star product algebra into $\mathcal{L}(\mathcal{H}_D \otimes \mathcal{H}_{D'})$ (and this is an injection). The H-invariance means that the factor in $\mathcal{H}_{D'}$ must be an intertwining operator for D'. Since this is irreducible this factor must be a multiple of the identity, so that the image of $\mathcal{S}_D$ under Weyl quantization is contained in $\mathcal{L}(\mathcal{H}_D) \otimes 1$. On the other hand the functions $S_g * p_D$ are H-invariant and it follows from the above observations that

$$W(S_g * p_D) = U(g)P_D = (D(g) \otimes 1)P_D$$

on $\mathcal{H}_D$. These operators generate the entire algebra $D(G)'' \cong \mathcal{L}(\mathcal{H}_D)$, so the proof is complete. □

When the compactness assumption is dropped one gets a similar result but can deduce only that one has a dense subalgebra of $\mathcal{L}(\mathcal{H}_D)$.

It is not immediately obvious how the $*$-algebra so constructed is related to an algebra of functions on the orbit. However, we know that in the limit as $\hbar \to 0$ the star product becomes ordinary multiplication, $\mathcal{I}_D$ is precisely the ideal of functions vanishing on $\mathcal{O}$ and the quotient algebra is the algebra of functions on $\mathcal{O}$. In this case $\Phi_D^* : f \mapsto \mathcal{F}p_D \star \Phi^* f$ defines an isomorphism and, since everything depends continuously on $\hbar$, and singularity is a closed condition on an operator on a finite-dimensional space, the same must be true for all $\hbar$ in some neighbourhood of 0. The star product can now be pulled back to $\mathcal{S}(h^*)$ using the product in $\mathcal{S}(V)$, that is

$$(\Phi_D^* \phi) \star (\Phi_D^* \psi) = \Phi_D^*(\phi \star \psi).$$

This is possible since the star product of H-invariant functions on the left is H-invariant, and similarly is invariant under multiplication by $\mathcal{F}p_D$, so by isomorphism it must correspond to some unique element in the image of Φ_D^*. It is the fact that Φ_D^* depends on the quadratic map $\Phi : V \to \mathfrak{g}^*$ which makes the star products look so complicated.

There is a slightly different way of describing the non-linear star products without specifying any particular representation D. The Peter–Weyl theorem tells us that for compact groups G the group algebra $L^1(G)$ is the direct sum of all the matrix algebras $\mathcal{L}(\mathcal{H}_D)$. Not all irreducibles appear in the metaplectic representation, but the map taking $F \in C(G)$ to

$$\int_G F(g) S_g \,\mathrm{d}g$$

gives an explicit homomorphism from $L^1(G)$ to H-invariant on V. We can therefore deal with all the irreducibles D simultaneously simply using $\mathcal{S}^H(V)$ with the induced star product.

10. Berezin quantization

Ideally we should like to complete the above discussion of non-linear star product algebra $\mathcal{S}^H(V)/\mathcal{I}_D$ with a description of the exponential functions. Unfortunately, although it is clear that these should be the functions $E_g^D = \mathcal{F}(p_D * S_g)$, these are not correctly normalized, since $E_1^D = \mathcal{F}p_D$ is not constant. There are various possible responses to this. We could, for example, relax the normalization condition to say that star multiplication by E_1^D should act as the identity, which is true by the definition of p_D. One interesting alternative, however closer to Fronsdal's approach and links directly to much of the recent work, is to use Berezin quantization. This also has the advantage of giving explicit formulae for the functions on a G-orbit. We shall assume in this and the next section that G is compact.

Suppose that the orbit $\mathcal{O} \cong G/K$, where K is a subgroup of G containing $G \cap H$. We first note that any choice of a density operator $\rho \in \mathcal{L}(\mathcal{H}_D)$ which commutes with the action of $D(K)$ defines an embedding of the matrix algebra $\mathcal{L}(\mathcal{H}_D) = D'(H)'$ into the space of functions on G/K by taking the operator $A \in \mathcal{L}(\mathcal{H}_D)$ to its Berezin symbol, the function

$$f_A(xK) = \mathrm{tr}(D(x)\rho D(x)^{-1}A),$$

[Be1,Be2]. (When ρ is the projection onto a vector Ω, $f_A(xK) = \langle D(x)\Omega, AD(x)\Omega\rangle$, whilst $D(x)\Omega/\langle\Omega, D(x)\Omega\rangle$ is the reproducing kernel.) This expression can be simplified by introducing the adjoint representation $\mathrm{Ad}_D(g)A = D(g)AD(g)^{-1}$ on $A \in \mathcal{L}(\mathcal{H}_D)$ with the Hilbert–Schmidt (trace) inner product, so that the identity reduces to

$$f_A(xK) = \langle \mathrm{Ad}_D(x)\rho, A\rangle_{\mathrm{tr}}.$$

The role of ρ would be similar to a generalized coherent state [Pe1,Pe2,A] except that Ad_D is not irreducible. We can now generalize some of the well-known results to reconstruct A from its Berezin symbol.

Theorem 10.1. *Let S be a unitary representation of a compact group G on a finite-dimensional inner product space, and let $\{T_j\}$ be a basis for the operators which intertwine S, orthonormal with respect to the trace inner product. Then, with normalized Haar measure, for any vectors ϕ, ψ, ξ*

$$\int_G \langle\psi, S(g)^*\xi\rangle S(g)\phi \,\mathrm{d}g = \sum_j \langle T_j\psi, \phi\rangle T_j\xi.$$

Proof. We know that, since it intertwines S the integral can be decomposed in terms of the basis $\{T_j\}$:

$$\int_G S(g)\phi \otimes \psi^* S(g)^* \,\mathrm{d}g = \sum_j c_j T_j,$$

and the coefficient c_k is found by multiplying by T_k^* and taking the trace of the operators on the two sides. □

In the case where all the irreducible constituents of S are multiplicity-free the basis elements T_j can be chosen to be multiples of the projections onto the irreducibles. The theorem can also be generalized to square-integrable representations of non-compact groups provided one has some control over the intertwining algebra.

Corollary 10.2. *If, under the assumptions of Theorem 10.1, $\langle\psi, \mathcal{I}\phi\rangle = \mathrm{tr}(\mathcal{I})$, for all intertwining operators $\mathcal{I}$ then for all ξ*

$$\int_G \langle\psi, S(g)^*\xi\rangle S(g)\phi \,\mathrm{d}g = \xi.$$

If ϕ is cyclic for S then it is always possible to find a unique vector ψ of the form $C\phi$, with C and intertwining operator, for which this holds, and conversely if ψ is cyclic there is a unique ϕ of the form $\tilde{C}\psi$.

In the special case when the projections P_j onto the irreducible components (of dimension d_j) of Ad_D are multiplicity free, one has the explicit formulae

$$\psi = \sum_j \frac{d_j}{\langle\phi, P_j(\phi,\phi)\rangle} P_j(\phi), \qquad \phi = \sum_j \frac{d_j}{\langle\psi, P_j(\psi)\rangle} P_j(\psi).$$

Proof. The identity is an intertwining operator and so can be expanded in terms of the basis as $1 = \sum_j \mathrm{tr}(T_j^*)T_j$, so that $\xi = \sum_j \mathrm{tr}(T_j^*)T_j\xi$. Comparing this with Theorem 10.1 we see that

$$\int_G \langle\psi, S(g)^*\xi\rangle S(g)\phi \,\mathrm{d}g = \xi$$

for all ξ if and only if $\mathrm{tr}(T_j^*) = \langle T_j\psi, \phi\rangle = \langle\psi, T_j^*\phi\rangle$ for all j. Any intertwining operator is a linear combination of the T_j^*, so the first result follows. Trying

$$\psi = C\phi = \sum c_k T_k \phi$$

in the basis-dependent form, we see that the condition becomes

$$\sum \langle T_j\phi, T_k\phi\rangle_{\mathrm{tr}}\, c_k = \mathrm{tr}(T_j^*).$$

The matrix of coefficients $\langle T_j\phi, T_k\phi\rangle_{\mathrm{tr}}$ is positive, and, when ϕ is cyclic it is actually positive definite. (This is because, if it were singular, then we could find γ_k such that $\sum_k \langle T_j\phi, T_k\phi\rangle\gamma k = 0$. Multiplying by γ_j and summing this would give $\| \sum \gamma_k T_k\phi \|^2 = 0$, forcing

$$\sum \gamma_k T_k \phi = 0.$$

When ϕ is cyclic this gives

$$\sum \gamma_k T_k \xi = 0$$

for all ξ, forcing $\sum \gamma_k T_k = 0$. This is possible only for vanishing coefficients γ_k, since the T_j form a basis.) Consequently the components, c_k, and so C itself are uniquely defined, giving the stated result for ψ. The reverse derivation of ϕ from Q_0 proceeds similarly. For future reference we note that we may choose the T_j to be self-adjoint, and then the c_k are real, so that C is self-adjoint. Finally, when the irreducibles are multiplicity-free we may take $T_j = d_j^{-1/2} P_j$ and use the orthogonality of the projections to reduce the conditions on the c_k to

$$\langle \phi, P_j \phi \rangle c_j = d_j^{3/2},$$

from which the other formulae can be deduced. □

Corollary 10.3. *If G is compact and ρ is Ad_D-cyclic then there exists a unique operator Q_0 of the form C_ρ for which*

$$A = \int_{G/K} f_A(xK) \mathrm{Ad}_D(x) Q_0 \, \mathrm{d}xK.$$

Proof. We apply Corollary 10.2 to the representation $S = \mathrm{Ad}_D$ with $\phi = \rho, \xi = A$, and $Q_0 = \psi$, and use $f_A(xK) = \langle \mathrm{Ad}_D(x)\rho, A \rangle$. □

Corollary 10.3 gives us a direct method of reconstructing A from its symbol. This differs from the usual formulae, obtained by analytic continuation of the kernel of A off the diagonal, in that it is more explicit and needs no complex structure. In fact we also obtain a formula for the continuation as

$$\langle D(x)\rho D(y)^*, A \rangle = \int_{G/K} f_A(zK) \langle D(x)\rho D(y)^*, \mathrm{Ad}_D(z) Q_0 \rangle \, \mathrm{d}zK.$$

It also strongly suggests that we should define the quantization of any integrable function, f, on G/K to be

$$Q(f) = \int_{G/K} f(xK) \mathrm{Ad}_D(x) Q_0 \, \mathrm{d}xK.$$

(The Moyal quantization of [Mo5], though different in aim, uses a similar formula with the geodesic symmetry playing the role of Q_0.) This is a D-covariant quantization in the sense that $\mathrm{Ad}_D(g)Q(f) = D(g)Q(f)D(g)^{-1} = Q(g \cdot f)$ for all $g \in G$ and functions f on G/K, where, as usual, $(g \cdot f)(xK) = f(g^{-1}xK)$. With this definition we have $Q(f_A) = A$. We shall say that any linear map Q from the chosen space of functions on G/K to $\mathcal{L}(\mathcal{H}_D)$ is ρ-normal if $Q(f_A) = A$ (cf. [HaHe]).

Finally we just note that the cyclicity of ρ and Q_0 is really needed. If Q_0 were not cyclic then it would generate a subspace of $\mathcal{L}(\mathcal{H}_D)$, and we could only reconstruct the component of A in that subspace. If ρ were not cyclic then f_A would lose all information about the components of A not in the subspace generated by ρ. (This provides a motivation for Fronsdal's condition that his function space $\mathcal{T}(M)$ must contain the representation Ad_D on $\mathcal{L}(\mathcal{H}_D) \cong \mathcal{H}_D \otimes \mathcal{H}_D$ [F, Eq. (115)].

The most important consequence of the formula reconstructiong A from its symbol is that it enables us to reproduce the effect of operator products by multiplication of their symbols. (The usual approach is to compose their kernels which we could also do using the formula following corollary.)

Theorem 10.4. *The product of functions on G/K defined by*

$$(f \star g)(xK) = \int_{G/K \times G/K} f(yK)g(zK)\langle \mathrm{Ad}_D(x)\rho, \mathrm{Ad}_D(y)Q_0\mathrm{Ad}_D(z)Q_0\rangle \, \mathrm{d}yK \, \mathrm{d}zK$$

satisfies $f_A \star f_B = f_{AB}$.

Proof. We simply substitute the formulae for A and B in terms of their symbols into $f_{AB}(xK) = \langle \mathrm{Ad}_D(x)\rho, AB\rangle$ to get $f_{AB} = f_A \star f_B$. □

As usual for function spaces associated with coherent states there is a reproducing kernel.

Theorem 10.5. *When ρ and Q are related as Corollary 10.3, the function $k_y(xK) = \langle \mathrm{Ad}_D(x)Q_0, \mathrm{Ad}_D(y)\rho\rangle$ provides a reproducing kernel for the functions on G/K with respect to the L^2 inner product. This kernel may also be expressed as*

$$k_y(xK) = \langle \mathrm{Ad}_D(x)\rho, \mathrm{Ad}_D(y)Q_0\rangle = f_{\mathrm{Ad}_D(y)Q_0}.$$

Proof. Taking the inner product of the condition that Q be ρ-normal with $\mathrm{Ad}_D(y)\rho$ gives

$$\begin{aligned}
&\int_{G/K} \overline{k_y(xK)} f_A(xK) \, \mathrm{d}xK \\
&\quad = \int_{G/K} \langle \mathrm{Ad}_D(y)\rho, \mathrm{Ad}_D(x)Q_0\rangle_{\mathrm{tr}} f_A(xK) \, \mathrm{d}xK \\
&\quad = \langle \mathrm{Ad}_D(y)\rho, A\rangle_{\mathrm{tr}} = f_A(yK),
\end{aligned}$$

which is the reproducing property. Using the connection between ρ and Q_0 we also see that

$$\langle \mathrm{Ad}_D(x)\rho, \mathrm{Ad}_D(y)Q_0\rangle_{\mathrm{tr}} = \langle \mathrm{Ad}_D(x)\rho, \mathrm{Ad}_D(y)C\rho\rangle_{\mathrm{tr}} = \langle \mathrm{Ad}_D(x)C\rho, \mathrm{Ad}_D(y)\rho\rangle_{\mathrm{tr}},$$

the last step following from the self-adjointness and intertwining properties of C. □

11. Normal exponential maps and star products

Any Berezin quantization gives very explicit formulae for the relationship between functions on an orbit G/K and operators. The exponential functions are clearly given by

$$E_g^\rho(xK) = f_{D(g)}(xK) = \mathrm{tr}(D(x)\rho D(x)^{-1}D(g)) = \mathrm{tr}(\rho D(x^{-1}gx)).$$

Since ρ is a density operator it immediately follows that $E_1^\rho = \mathrm{tr}(\rho) = 1$, so that E_1^ρ is automatically normalized. Since D is irreducible, one of the components of Ad_D is the one-dimensional space of intertwining operators, which is projected by $P_1(A) = (\mathrm{tr}(A)/d_D)1$, where d_D is the dimension of D. All other components being orthogonal to this will have vanishing trace. Applying the Theorem 10.1 to

$$\begin{aligned}\int_G E_g^\rho(xK)\,\mathrm{d}xK &= \int_G \langle \mathrm{Ad}_D(x)\rho, D(g)\rangle_{\mathrm{tr}}\,\mathrm{d}xK\\ &= \int_G \langle \mathrm{Ad}_D(x)\rho, D(g)\rangle_{\mathrm{tr}} \mathrm{Ad}_D(x)(1)\,\mathrm{d}xK\end{aligned}$$

gives just a single component $\langle \rho, 1\rangle_{\mathrm{tr}} P_1(D(g)) = \mathrm{tr}(\rho)\mathrm{tr}(D(g))/d_D = \chi_D(g)/d_D$, which is Fronsdal's character formula. Since D is irreducible, every operator $A \in \mathcal{L}(\mathcal{H}_D)$ is in the span of the $D(g)$, so that the star product defined by

$$E_g^\rho \star E_h^\rho = E_{gh}^\rho$$

(the multiplier α being trivial on the compact subgroup G) will automatically aslo satisfy

$$f_A \star f_B = f_{AB}.$$

One can also obtain Fronsdal's expression for the exponential map in terms of spherical harmonics by choosing orthonormal bases T_m^j for $P_j\mathcal{L}(\mathcal{H}_D)$ and then noting that

$$E_g^\rho(xK) = \langle \mathrm{Ad}_D(x)\rho, D(g)\rangle_{\mathrm{tr}} = \sum \langle \mathrm{Ad}_D(x)\rho, T_m^j\rangle_{\mathrm{tr}}\langle T_m^j, D(g)\rangle_{\mathrm{tr}}.$$

(Fronsdal's formula uses for ρ the projection onto a highest weight vector for K.)

Finally, let us see how the Berezin quantization arises in the context of Section 13. There certainly exists a tempered distribution ρ_0 such that $W(\rho_0) = \rho$. Moreover, the fact that ρ is associated with D means that $p_D * \rho_0 = \rho_0$, from which we deduce that ρ_0 is actually a function. Exploiting the self-adjointness of ρ, we may therefore write

$$\begin{aligned}E_g^\rho(xK) &= \mathrm{tr}(\rho^* D(x^{-1}gx)) = \mathrm{tr}(W(\rho_0)^* P_D U(x^{-1}gx))\\ &= \mathrm{tr}(U(x)W(\rho_0)^* U(x^{-1})U(g)) = \int_V \overline{(x\cdot\rho_0)(v)}\mathrm{tr}(W(v)^* U(g))\\ &= |\Lambda|^{-1}\int_V \overline{(x\cdot\rho_0)(v)} S_g(v)\,\mathrm{d}v = |\Lambda|^{-1}\langle x\cdot\rho, S_g\rangle.\end{aligned}$$

(For future reference, we note that we can similarly obtain the useful identity

$$\langle \mathrm{Ad}_D(x)\rho, W(f)\rangle_{\mathrm{tr}} = |\Lambda|^{-1}\langle x\cdot\rho_0, f\rangle$$

for any $f \in \mathcal{S}^H(V)$.) To make the connection with our previous notions we Fourier-transform this, and to simplify the notation we use hats to denote transforms, so that $\hat{f} =$

$\mathcal{F}f$. Applying Plancherel's theorem $|\Lambda|^2\langle\hat{f}, \hat{g}\rangle = \langle f, g\rangle$, we deduce the first part of the following theorem.

Theorem 11.1. *The normal exponential map is related to that for the linear problem by*

$$E_g^\rho(xK) = |\Lambda|\langle x \cdot \hat{\rho}_0, E_g\rangle = |\Lambda| \int_V \overline{(x \cdot \hat{\rho}_0)(v)} E_g(v)\, \mathrm{d}v.$$

The map taking $\phi \in \hat{p}_D \star \mathcal{S}^H(V)$ to

$$\phi^\rho(xK) = |\Lambda|\langle x \cdot \hat{\rho}_0, \phi\rangle = |\Lambda| \int_V \overline{(x \cdot \hat{\rho}_0)(v)} \phi(v)\, \mathrm{d}v$$

is a homomorphism of star product algebras.

Proof. The second part follows from the fact that the exponential maps generate each algebra, and satisfy the same product rule. □

Since $x \cdot \rho_0 * p_D = x \cdot \rho_0$, we could equally well replace E_g by E_g^D. We can also give an explicit formula for the inverse of this homomorphism in terms of the Berezin quantization. For a start there must similarly be a function $q_0 \in p_D \star \mathcal{S}^H(V)$ such that $Q_0 = W(q_0)$, the relationship between q_0 and ρ_0 being identical to that between Q_0 and ρ:

$$q_0 = \sum_j \frac{d_j}{\langle \rho_0, (\rho_0)_j\rangle} (\rho_0)_j,$$

where $(\rho_0)_j$ is given in terms of the character, χ_j, of the irreducible component of Ad_D as

$$(\rho_0)_j = \int_G \overline{\chi_j(g)} (g \cdot \rho)\, \mathrm{d}g.$$

Now we know that a function f on G/K has a Berezin quantization as

$$Q(f) = \int_{G/K} f(xK) \mathrm{Ad}_D(x) Q_0\, \mathrm{d}xK = \int_{G/K} f(xK) W(x \cdot q_0)\, \mathrm{d}xK,$$

which is the Weyl quantization of

$$\int_{G/K} f(xK) x \cdot \hat{q}_0\, \mathrm{d}xK.$$

Thus we have the following result.

Theorem 11.2. *The inverse to the map $\phi \mapsto \phi^\rho$ is the map*

$$f \mapsto \int_{G/K} f(xK) x \cdot \hat{q}_0\, \mathrm{d}xK.$$

We may easily check from this that, in particular, $k_y = (y \cdot q_0)^\rho$. The advantage of this correspondence is that the star product is much more readily described in $S^H(V)$ than directly in terms of functions on G/K, and we can now exploit the correspondence to obtain an integral formula for the product.

Theorem 11.3. *The product of functions f and g on G/K is given by*

$$(f \star g)(xK) = |\Lambda|^{-1} \int_{G/K\times G/K} f(yK)g(zK)\langle x \cdot \rho_0, y \cdot q_0 * z \cdot q_0\rangle \, \mathrm{d}yK \, \mathrm{d}zK.$$

Proof. This follows from the fact that Berezin quantization is a homomorphism, so that

$$\begin{aligned}(f \star g)(xH) &= \langle x \cdot \rho, Q(f)Q(g)\rangle_{\mathrm{tr}} \\ &= \int_{G/K\times G/K} f(yK)g(zK)\langle x \cdot \rho, W(y \cdot q_0)W(z \cdot q_0)\rangle_{\mathrm{tr}} \, \mathrm{d}yK \, \mathrm{d}zK \\ &= |\Lambda|^{-1} \int_{G/K\times G/K} f(yK)g(zK)\langle x \cdot \rho_0, (y \cdot q_0) * (z \cdot q_0)\rangle \, \mathrm{d}yK \, \mathrm{d}zK,\end{aligned}$$

as required. □

12. Examples

The first example to which one always turns is that of $SU(2)$. In the present context it means considering the dual pair $G = U(2)$, $H = U(1)$ in $Sp(\mathbb{R}^4)$. We take the subgroup K of diagonal matrices in $U(2)$, so that the quotient G/K is a sphere, S^2. When $D = D^l$ we have

$$D^l \otimes D^{l*} \cong D^{2l} \oplus \cdots \oplus D^0,$$

which is multiplicity-free, so that the projections P_j in Ad_D may be used as a basis for the intertwiners and they will pick out the tensor operators transforming with D^j for some integral $j \leq 2l$, and $d_j = (2j+1)$. Since ρ commutes with the action of K we see that $P_j(\rho)$ must be the weight zero component of the tensor operator. By the general Wigner–Eckart theorem, we therefore have

$$\mathrm{tr}(\rho \mathrm{Ad}_D(x)P_j(\rho)) = \langle e_0, D^j(x)e_0\rangle \mathrm{tr}(\rho P_j(\rho)),$$

where e_0 is a weight zero vector in the representation space of D^j. This matrix element is given in terms of a Legendre polynomial, $\mathcal{P}_j$. Our general formula for the reproducing kernel therefore gives, for $\boldsymbol{x}, \boldsymbol{y}$ in the unit sphere,

$$k_y(\boldsymbol{x}) = \sum(2j+1)\mathcal{P}_j(\boldsymbol{y} \cdot \boldsymbol{x}).$$

As a matter of fact we could have found this without calculation, since it is known that $(2j+1)\mathcal{P}_j(\boldsymbol{y}\cdot\boldsymbol{x})$ is the (appropriately normalized) reproducing kernel for the subspace of $L^2(S^2)$ transforming with D^j.

It is also quite easy to find exponential functions. For this we shall make the particular choice of ρ as the projection onto a highest weight vector, Ω, in the space of D^l. Then

$$E_g^\rho(K) = \langle \Omega, D^l(g)\Omega\rangle.$$

But since D^l can be realized as the $(2l)$th symmetric power of natural representation and Ω as the tensor power of its highest weight vector, e_+, we have

$$E_g^\rho(K) = \langle e_+, ge_+\rangle^{2l}.$$

When g represents a rotation through an angle θ about an axis $\boldsymbol{n}$ and $\boldsymbol{k}$ is a vector stabilized by K, this gives

$$E_g^\rho(K) = (\cos\tfrac{1}{2}\theta + \mathrm{i}\sin\tfrac{1}{2}\theta\boldsymbol{n}\cdot\boldsymbol{k})^{2l}.$$

We therefore deduce that in general

$$E_g^\rho(\boldsymbol{x}) = (\cos\tfrac{1}{2}\theta + \mathrm{i}\sin\tfrac{1}{2}\theta\boldsymbol{n}\cdot\boldsymbol{x})^{2l}.$$

This can also be written in a more invariant form as

$$E_g^\rho(\boldsymbol{x})[\mathrm{tr}(\tfrac{1}{2}(1+\sigma\cdot\boldsymbol{x})g)]^{2l},$$

(compare [Mo2, Section 4]).

13. Asymptotic behaviour of star products

Star products are supposed to be deformations of pointwise multiplication of functions, but, when dealing with systems symmetric under a simple Lie group, there are no obvious deformation parameters to hand, so one adopts a slightly different approach. In our formulation one considers a family of density operators $\rho^{(k)}$, which are the projections onto the k-fold tensor powers, $\Omega^{(k)}$, of a vector Ω in the representation space of D. (When D is realized on sections of a line bundle $\mathcal{L}$, whose fibre at xK is spanned by $D(x)\Omega$, then $D^{(k)}(x)\Omega^{(k)}$ spans the fibre of $\mathcal{L}^k$.) The symmetric tensor product $D^{(k)}$ acts on the space containing $\Omega^{(k)}$, and there is a convenient nesting property that, for all $k > l$, the representation $\mathrm{Ad}_{D(k)}$ contains $\mathrm{Ad}_{D(l)}$. This can be proved by noting the obvious injection $i : \mathcal{L}(\mathcal{H}^{(k-1)}) \mapsto \mathcal{L}(\mathcal{H}^{(k-1)} \otimes_S 1 \subseteq \mathcal{L}(\mathcal{H}^{(k)})$. (This observation, which can also be derived by a duality argument, helps one to find the intertwining operators for $\mathrm{Ad}_{D^{(k)}}$). If one reconstructs an operator on the tensor power using f_A, for A of order r, that is in $\mathcal{L}(\mathcal{H}^{(r)})$, then one actually gets $i^{k-r}(A) = A \otimes_S 1^{(k-r)}$, since

$$\langle \mathrm{Ad}_{D^{(k)}}(g)\rho^{(k)}, A \otimes_S 1^{(k-r)}\rangle = \langle \mathrm{Ad}D^{(r)}(g)\rho^{(r)}, A\rangle = f_A(gK).$$

Suppose now that we want to calculate the star product $f_A \star_k f_B$ for B an operator on $\mathcal{H}^{(s)}$, using $\rho^{(k)}$. Since f_A and f_B are also the symbols of the operators $A \otimes_S 1^{(k-r)}$ and $B \otimes 1^{(k-s)}$, we get the symbol of their product $(A \otimes_S 1^{(k-r)})(B \otimes 1^{(k-s)})$.

Consider the first case $r = s = 1$, when we have

$$\begin{aligned}
&(A \otimes_S 1^{(k-r)})(B \otimes_S \delta 1^{(k-s)})\xi^{(k)} \\
&\quad = (A \otimes_S 1^{(k-r)})(B\xi \otimes_S \xi^{(k-1)}) \\
&\quad = \frac{1}{k}(AB\xi \otimes \xi^{(k-1)} + (k-1)A\xi \otimes_S B\xi \otimes_S \xi^{(k-2)}),
\end{aligned}$$

since in the symmetrized product A must act on each term and it usually misses the term with the B. Now

$$\begin{aligned}
\langle \mathrm{Ad}_{D^{(k)}}(g)\rho^{(k)}, A \otimes_S B \otimes_S 1^{(k-2)} \rangle &= \langle \mathrm{Ad}_D(g)\rho, A\rangle \langle \mathrm{Ad}_D(g)\rho, B\rangle \\
&= f_A(gK) f_B(gK),
\end{aligned}$$

so that the net result is

$$f_A \star_k f_B = \frac{1}{k} f_{AB} + \frac{k-1}{k} f_A f_B.$$

Viewed this way the fact that the star product has the usual product as an asymptotic limit is just a cluster property of the state $\rho^{(k)}$ of a kind familiar from quantum field theory.

This easily generalizes to give the following theorem.

Theorem 13.1. *Let A be of order r and B of order s, and let $f_A \circ_p f_B$ denote the symbol of the terms in the composition of $i^{k-r}(A)$ and $i^{k-s}(B)$ in which p factors in the tensor product are acted on by both A and B. Then*

$$f_A \star_k f_B = \sum_{p=\max(0,r+s-k)}^{\min(r,s)} \frac{(k-r)!(k-s)!}{k!(k+p-r-s)!} f_A \circ_p f_B,$$

where $f_A \circ_0 f_B = f_A f_B$ is the pointwise product.

Proof. The proportion of terms for which the action of A and B overlaps on just p common factors follows a hypergeometric distribution, and is given by

$$(k-r)!(k-s)!r!s!/k!(k+p-r-s)!(r-p)!(s-p)!p!.$$

A proportion $\binom{r}{p}\binom{s}{p}p! = r!s!/(r-p)!(s-p)!p!$ of this is accounted for by the ways of choosing the p common elements from A and B, and of pairing them, which we absorb into the expression $f_A \circ_p f_B$. On doing the division one obtains the coefficient as given. □

One deduces immediately that for $k > r, s$, $f_A \star_k f_B$ is a rational function of k which is regular at $k = \infty$. (In [Mo1–6,CRG1–3]), results of this sort are derived in a more general context, though with a less explicit or different form of expansion.) Using Stirling's formula we see that the coefficient of $f_A \star_k f_B$ is of order k^{-p}. The top term is just a multiple of the ordinary product,

$$[(k-r)!(k-s)!/k!(k-r-s)!]f_A f_B \sim (1-rs/k)f_A f_B,$$

and it is easy to relate the skew part of the first-order term to the Poisson bracket $\{f_A, f_B\}$. To allow for the wide variety of normalization convections in common use we shall simply introduce a constant κ.

Corollary 13.2. *Under the previous assumptions, the leading term in $f_A \star_k f_B - f_B \star_k f_A$ is*

$$\mathrm{i}\kappa \frac{(k-r)!(k-s)!}{k!(k+1-r-s)!}\{f_A, f_B\}.$$

Proof. We first consider the case of $r = s = 1$, In this case A and B are quadratic functions, for which it is known that the quantization is exact, that is

$$f_A \star_1 f_B - f_B \star_1 f_A = \mathrm{i}\kappa\{f_A, f_B\}.$$

Next consider the case of factorizable A and B, that is

$$A = A_1 \otimes_S A_2 \otimes_S \cdots \otimes_S A_r, \qquad B = B_1 \otimes_S B_2 \otimes_S \cdots \otimes_S B_s$$

with $f_A = f_{A_1}, \ldots, f_{A_r}$, and $f_B = f_{B_1}, \ldots, f_{B_s}$. As usual the pointwise product terms in $f_A \star_k f_B$ and $f_B \star_k f_A$ cancel, leaving the leading $p = 1$ term

$$\frac{(k-r)!(k-s)!}{k!(k+1-r-s)!}(f_A \circ_1 f_B - f_B \circ_1 f_A).$$

Now $f_A \circ_1 f_B$ contains the terms where just one of the A and B factors overlap, A_i and B_j, say. The difference $f_A \circ_1 f_B - f_B \circ_1 f_A$ is thus a sum of terms of the form

$$\prod_{m\neq i} f_{A_m} \prod_{n\neq j} f_{B_n}(f_{A_i} \circ_1 f_{B_j} - f_{B_j} \circ_1 f_{A_i}) = \mathrm{i}\kappa \prod_{m\neq i} f_{A_m} \prod_{n\neq j} f_{B_n}\{f_{A_i}, f_{B_j}\}.$$

By the Leibniz property of the Poisson bracket this reduces to $\mathrm{i}\kappa\{f_A, f_B\}$, as asserted. □

When dealing with Kähler manifolds

$$\frac{f_A(z)}{f_{\rho^{(r)}}(z)} = \frac{\langle D^{(r)}(z)\Omega^{(r)}, AD^{(r)}(z)\Omega^{(r)}\rangle}{|\langle\Omega^{(r)}, D^{(r)}(z)\Omega^{(r)}\rangle|^2}$$

can be regarded as holomorphic function of z and $\bar{z}$. Recalling that $f_{\rho^{(r)}} = f_\rho^r$, the Poisson bracket can be written as

$$\{f, g\} = \sum_j (\overline{\nabla} f \nabla_j g - \overline{\nabla}_j g \nabla_j f),$$

where ∇_j acts as $f_\rho^{1/2-r}\partial_j f_\rho^r$, and ∂_j denotes the derivative with respect to a homogeneous complex coordinate. The higher-order terms in the expansion can be likewise be written in terms of $\overline{\nabla}_j^p \nabla_j^p$, which links in with the results of Moreno, Cahen, Gutt and Rawnsley. For example, in the simplest case of $r = s = 1$ on $\mathbb{CP}^n$, with the representative point $(1, z)$, for $z \in \mathbb{C}^n \cup \infty$, we readily calculate that

$$f \star_k g = fg + \frac{1+ \| z \|^2}{k} \frac{\partial f}{\partial z_j} (\delta_{jk} + z_j \bar{z}_k) \frac{\partial g}{\partial \bar{z}_k}.$$

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